

A VIEW OF 2-DIM. STRING THEORY AND BLACK HOLES¹

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Abstract

We present a brief overview of 2-dim. string theory and its connection to the theory of non-relativistic fermions in one dimension. We emphasize (i) the role of W_∞ algebra and (ii) the modelling of some aspects of 2-dim. black hole physics using the phase space representation of the fermi fluid.

0. Introduction

There are many issues of physical interest in string theory that require a non-perturbative formulation; for example black holes, supersymmetry breaking, quark lepton masses are a few of the phenomena of interest. Presently such a formulation is not available to us in 4 dimensions. Not only do we not know the laws of string theory in 4-dim. but we do not even know its true microscopic degrees of freedom. It is in these circumstances that toy models become important because their formulation and solution may shed some light on the more realistic issues of string theory which we would like to address.

An important model that has been much studied is 2-dim. string theory. There are several reasons for this:

- (i) Firstly 2-dim. string theory is non-trivial in content. Its weak coupling (low energy) spectrum has a massless particle. There is a non-trivial S -matrix of the massless particles. Besides that it also has a discrete infinity of backgrounds which are the remnants of the massive string modes in higher dimensions.
- (ii) The low energy limit (in the σ -model approach) has a black hole solution which is characterized by a mass.
- (iii) There is a matrix model formulation of the theory that in principle defines the theory non-perturbatively for all values of the coupling constant. The matrix model is exactly formulated as a system of non-relativistic fermions in 1-dimension (the full real line) in an inverted harmonic oscillator potential.

¹Based on the talk given at the 'International Colloquium on Modern Quantum Field Theory', Bombay, January 1994

- (iv) The matrix model formulation has several important implications:
- a) Firstly it enables computations which can be performed to all orders in perturbation theory.
 - b) It indicates in a simple way the existence of non-perturbative effects that go as $e^{-\frac{1}{g_{str}}}$ which are characteristic of string theory.
 - c) There are an infinite number of conservation laws. The phase space of the theory is characterized as a non-linear representation of W_∞ algebra.
 - d) There is a real possibility of studying non-perturbative aspects of black holes.

1. The Matrix Model Formulation of 2-dim. String Theory

In the following we briefly spell out the matrix model formulation and indicate the connections with the target space.

As is well known a discrete formulation of 2-dim. gravity leads to the matrix model[1]. The 1-dim. matrix model describes the coupling of 1-dim. matter to 2-dim. gravity. The Liouville mode of 2-dim. gravity can be identified with a space co-ordinate[2] and hence this model is in fact a string theory in a 2-dim. target space-time. It is in this way that the matrix model describes 2-dim. string theory. The matrix model in turn maps into the problem of non-relativistic fermions in 1-dim.[3] In the double scaling limit the non-interacting fermions move in an inverted harmonic oscillator potential.[4] We can write the action in terms of the non-relativistic fermion field $\psi(x, t)$:

$$\begin{aligned}
S &= \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dx \psi^+(x, t) (i\partial_t - h_x) \psi(x, t) \\
h_x &= \frac{1}{2} \left(-\partial_x^2 + V(x) \right) \\
V(x) &= -x^2 + \frac{g_3 x^3}{\sqrt{N}} + \dots \\
&\int_{-\infty}^{+\infty} \psi^+ \psi dx = N.
\end{aligned} \tag{1}$$

The double scaling limit corresponds to $N \rightarrow \infty$ and the bare Fermi level (measured from the maximum of the potential) $\epsilon_F \rightarrow 0$, while keeping $\mu = N\epsilon_F$ fixed.

It is natural to identify the fermions as the microscopic degrees of freedom of the string theory. Since fermion number is held fixed, the physical variables are those which are invariant under the $U(1)$ transformation $\psi(x, t) \rightarrow e^{i\Theta} \psi(x, t)$. A general set of variables with this property are the bilocal variables[5, 6]:

$$\phi(x, y, t) = \psi(x, t) \psi^+(y, t) \tag{2}$$

They satisfy the W_∞ algebra[5, 6, 7, 8, 9]

$$\begin{aligned}
[\phi(x, y, t), \phi(x', y', t)] &= \phi(x' y, t) \delta(x - y') \\
&\quad - \phi(x, y', t) \delta(x' - y)
\end{aligned} \tag{3}$$

These are the Poisson brackets of our phase space. However the phase space is not linear and there are non-linear constraints reflecting the underlying fermion degree of freedom. Defining $\Phi(t)$ such that

$$\langle x|\Phi(t)|y\rangle = \phi(x, y, t) \quad (4)$$

the non-linear constraints are easily deduced using the fermion anti-commutation relations,

$$\begin{aligned} \Phi^2 &= \Phi \\ \text{tr}(1 - \Phi) &= N. \end{aligned} \quad (5)$$

The equation of motion is

$$i\partial_t \Phi + [\hat{h}, \Phi] = 0. \quad (6)$$

where \hat{h} is the single particle operator $\hat{h} = \frac{1}{2} \hat{p}^2 + V(\hat{x})$.

2. The Classical Fermi fluid

One can present an action principle and a path integral formulation for the system of equations (3),(4),(5). For details this we refer the reader to the published literature.[6, 10] Presently we express the above formulae in terms of the operator that describes the phase space distribution function of the fermions. It is basically a double fourier transform of the bilocal variable $\phi(x, y, t)$:

$$\hat{u}(p, q, t) = \int dx \psi^+(q - \frac{x}{2}, t) e^{-ipx} \psi\left(q + \frac{x}{2}, t\right) \quad (7)$$

Let us denote the classical phase-space distribution by $u(p, q, t)$. It is the expectation value of (7) in a W_∞ coherent state. It satisfies the Liouville equation (follows from (6))

$$\partial_t u + p\partial_q u + q\partial_p u = 0 \quad (8)$$

The constraints on u are (follows from (5)):

$$\left[\cos \frac{1}{2} (\partial_p \partial_{q'} - \partial_{p'} \partial_q) u(p, q, t) u(p', q', t) \right]_{\substack{p'=p \\ q'=q}} = u(p, q, t) \quad (9a)$$

$$\int \frac{dp dq}{2\pi} u(p, q, t) = N. \quad (9b)$$

Equations (8) and (9) are very difficult to solve. However one can discuss the hydrodynamic limit, where (9a) is replaced by the simpler constraint

$$u^2(p, q, t) = u(p, q, t) \quad (10)$$

which says that $u(p, q, t)$ is a characteristic function consistent with our idea of a classical fermi fluid.

Now consider a 2-dim. projection of the dynamics of the Liouville equation (8), by introducing the moments:

$$\begin{aligned}\rho(q, t) &= \int_{-\infty}^{+\infty} \frac{dp}{2\pi} u(p, q, t) \\ \pi(q, t) \rho(q, t) &= \int_{-\infty}^{+\infty} \frac{dp}{2\pi} p u(p, q, t) \\ \pi_2(q, t) \rho(q, t) &= \int_{-\infty}^{+\infty} \frac{dp}{2\pi} p^2 u(p, q, t) \text{ etc.}\end{aligned}\tag{11}$$

$\rho(q, t)$ and $\pi(q, t)$ correspond to the density and velocity of the fluid. The Liouville equation then corresponds to an infinite set of coupled equations in 2-dim. involving the functions ρ, π, π_2, \dots which are constrained by (10)[11],

$$\begin{aligned}\partial_t \rho + \partial_q(\rho \pi) &= 0. \\ \partial_t \pi &= \partial_q \left(\frac{\pi^2}{2} + \frac{q^2}{2} - \pi_2 \right) + \frac{\partial_q \rho}{\rho} \left(\pi^2 - \pi_2 \right), \text{ etc.}\end{aligned}\tag{12}$$

If we further assume that the characteristic functions are parameterized by ‘quadratic’ profiles which is a reasonable ansatz for describing very small ripples on the fermi surface:[12, 13, 14]

$$u(p, q, t) = \theta [(p_+(q, t) - p) (p - p_-(q, t))]\tag{13}$$

$p_{\pm}(q, t)$ parametrize the slightly deformed fermi surface in a way that conserves fermion number. Under the above approximation all the moments of $u(p, q, t)$ depend only on the first two moments $\rho(q, t)$ and $\pi(q, t)$, e.g. $\pi_2 = \pi^2 + \frac{\pi^2}{3} \rho^2$. Substituting in (12) we get the closed set of field equations:

$$\begin{aligned}\partial_t \rho + \partial_q(\pi \rho) &= 0 \\ \partial_t \pi + \pi \partial_q \pi &= -\partial_q \left(-\frac{q^2}{2} + \frac{\pi^2}{2} \rho^2 \right)\end{aligned}\tag{14}$$

The above are the hydrodynamic equations of a classical fermi fluid with density $\rho(q, t)$, velocity $\pi(q, t)$ and pressure $P(q, t) = -\frac{q^2}{2} + \frac{\pi^2}{2} \rho^2(q, t)$. They were originally obtained using collective field theory[12] rather than the bosonization we have done. Our method also brings out the fact that collective field theory is an approximate but useful hydrodynamic limit of the true bosonization of non-relativistic fermions.

The density and velocity satisfy the natural current commutation relations: (which can be derived from the W_{∞} commutation algebra of $u(p, q, t)$)

$$[\rho(q, t), \pi(q', t)] = i \partial_q \delta(q - q')\tag{15}$$

Using (15) one can see that the equation (14) follow from the action:

$$\begin{aligned}S &= \int dt dq \left(\pi \frac{1}{\partial_q} \dot{\rho} - H(\rho, \pi) \right) \\ H(\rho, \pi) &= \int dq \left[\frac{1}{2} \rho \left(\pi^2 + \frac{1}{3} \rho^2 \right) + \rho(V(q) + \mu) \right]\end{aligned}\tag{16}$$

where μ is the lagrange multiplier corresponding to the constraint $\int dq\rho = N$. We can consider doing perturbation theory in the action (16) around the filled fermi sea where the density is $\rho_0(q) = \sqrt{q^2 + 2\mu}$, ($\mu < 0$):

$$\rho(q, t) = \rho_0(q, t) + \partial_q \eta(q, t).$$

Then introducing the ‘time of flight’ variable τ , by the relation $\sqrt{\mu} \cosh \tau = q$ we get the action (16) in the chiral form.[13] (In ref. 13 this was obtained by computing correlation functions using the underlying fermion theory.)

$$\begin{aligned} S &= \int d\tau dt \left(\partial_+ \eta \partial_- \eta - \frac{1}{\mu} \frac{1}{\sinh^2 \tau} \left[(\partial_+ \eta)^3 - (\partial_- \eta)^3 \right] + \dots \right) \\ \partial_{\pm} &= \partial_{\tau} \pm \partial_t. \end{aligned} \quad (17)$$

The range of τ in (17) is $(0, -\infty)$, and the boundary condition $\eta(0, t) = 0$ follows from (9b). The action (17) is ill defined as the interaction diverges as $\tau \rightarrow 0$, which is the turning point of the classical motion. We have the problem of “wall scattering”. Using standard WKB methods we can replace (17) by an action in which the range of τ is $(-\infty + i\epsilon, +\infty + i\epsilon)$, $\epsilon > 0$. In fact ϵ can be chosen to be $\frac{\pi}{2}$ and the S -matrix can be calculated from the action

$$S = \int_{-\infty}^{+\infty} d\tau dt \left(\partial_+ \eta \partial_- \eta - \frac{1}{\mu} \frac{1}{\cosh^2 \tau} \left[(\partial_+ \eta)^3 - (\partial_- \eta)^3 \right] + \dots \right) \quad (18)$$

and then reinterpreted for “wall scattering”. [15] We remark that (18) has a natural interpretation in the momentum representation on non-relativistic fermions, which naturally leads to a non-singular interaction at $\tau = 0$. For details we refer to ref. 15.

3. The S -matrix

From (18) we can calculate the scattering amplitude of 4-massless particles of energies E_i [14, 15, 16, 17]

$$\begin{aligned} S(1, 2, 3, 4) &\propto \delta(E_1 + E_2 + E_3 + E_4) A(E_1, E_2, E_3, E_4) \\ A(E_1, E_2, E_3, E_4) &= \frac{1}{\mu^2} (|E_1 + E_2| + |E_1 + E_3| + |E_1 + E_4| - i) \end{aligned} \quad (19)$$

In the above we have used the mass shell conditions: $E_i + k_i = 0$.

Now we can interpret the above amplitude as a “wall scattering” process. For example consider the “3 \rightarrow 1” process, in which we scatter 3-particles at the “wall” and one comes back. Let us denote the momenta by w_i with the identification $w_1 \rightarrow -E_1, w_2 \rightarrow -E_2, w_3 \rightarrow -E_3$ and $w_4 \rightarrow E_4$, then the momentum non-conserving “wall scattering” amplitude is given by

$$\bar{A}(1 + 2 + 3 \rightarrow 4) \propto \frac{1}{\mu^2} (|w_1 + w_2| + |w_1 + w_3| + |w_1 - w_4| - i). \quad (20)$$

Connection with the “wall scattering” amplitude of 2-dim. string theory can be made by the suitable multiplication of (20) by leg pole factors: [18, 19]

$$\ell_{\text{in}}(w) = \left(\frac{\pi}{2} \right)^{i\frac{w}{4}} \frac{\Gamma(iw)}{\Gamma(-iw)}$$

for incoming particles and

$$\ell_{\text{out}}(w) = \left(\frac{\pi}{2}\right)^{-i\frac{w}{4}} \frac{\Gamma(-iw)}{\Gamma(iw)}$$

for out-going particles. The leg-pole factors have poles at imaginary momenta: $w = -in$, reflecting the “discrete states” of the 2-dim. string theory.[20, 21]

Hence using (20) the scattering amplitude for the process $1 + 2 + 3 \rightarrow 4$ is

$$A(1 + 2 + 3 \rightarrow 4) = \left(\frac{\pi}{2}\right)^{-\frac{iw_4}{2}} \prod_{i=1}^4 \frac{\Gamma(iw_i)}{\Gamma(-iw_i)} \cdot \bar{A}(1 + 2 + 3 \rightarrow 4). \quad (21)$$

In a similar fashion one can obtain the amplitudes $A(1 + 2 \rightarrow 3 + 4)$, $A(1 \rightarrow 2 + 3 + 4)$ etc.

It is interesting to note that the scattering amplitudes can also be derived directly from the W_∞ conservation laws.[22, 18] Let us briefly mention these conservation laws which easily follow from Liouville’s equation (8):

$$W_{rs} = \int \frac{dpdq}{2\pi} e^{-t(s-r)} (p-q)^r (p+q)^s u(p, r, t). \quad (22)$$

One can check that $\frac{d}{dt} W_{rs} = 0$. The amplitude $\bar{A}(1 + 2 + 3 \rightarrow 4)$ can be deduced by evaluating the charge W_{40} at times $t \rightarrow \pm\infty$, and then by obtaining a relation between the incoming ($t \rightarrow -\infty$) and outgoing waves ($t \rightarrow +\infty$). The conserved charges (22) are the classical limit of the exactly conserved operators in which the integrand $(p-q)^r (p+q)^s$ is replaced by phase space function corresponding to the Weyl-ordered single particle operator : $(\hat{p} - \hat{q})^r (\hat{p} + \hat{q})^s$:, where :: stands for Weyl ordering. Hence the conserved operators get corrections in powers of g_{str} . [6] These quantum conserved charges can presumably be used to compute corrections to the S -matrix in powers of the string coupling.

4. Beyond Scattering Amplitudes: Classical Solutions

In the previous section we outlined a procedure to obtain all tree level scattering amplitudes of the 2-dim. string theory in which the external states are the massless tachyons. However it is always more fruitful to have a classical action from where we can derive the scattering amplitudes. But it is not very easy to write down such an action as the scattering amplitudes only involve the massless tachyon. A different approach is to start with the continuum string theory and use the σ -model approach. In that approach it is natural to introduce vertex operators that correspond to not only the tachyon but also the graviton, dilaton etc.

The σ -model is described by the lagrangian

$$\mathcal{S} = \frac{1}{8\pi} \int d^2\xi \left(\frac{1}{2} \sqrt{\hat{g}} \hat{g}^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu - 2\hat{R}^{(2)} \Phi(x^\mu) + T(x^\mu) + \dots \right) \quad (23)$$

$G_{\mu\nu}$, Φ and T correspond to the graviton, dilaton and tachyon respectively. We will consider a truncated σ -model and presently ignore the other higher tensor fields. \hat{g}_{ab}

is a fiducial 2-dim. metric. The standard equations of motion at one-loop follow from the requirement of Weyl invariance: $\hat{g}_{ab} \rightarrow e^\sigma \hat{g}_{ab}$ [23]

$$\begin{aligned} R_{\mu\nu} - 2\nabla_\mu \nabla_\nu \Phi + \nabla_\mu T \nabla_\nu T &= 0 \\ R + 4(\nabla\Phi)^2 - 4\nabla^2\Phi + (\nabla T)^2 + V(T) - 4 &= 0. \\ -2\nabla^2 T + 4\nabla\Phi \cdot \nabla T + V'(T) &= 0 \\ V(T) &= -T^2 + \lambda T^3, \end{aligned} \tag{24}$$

It is easy to see that these equations follow from the action corresponding to 2-dim. dilaton-gravity coupled to T :

$$S = \int d^2x e^{-2\Phi} \sqrt{G} \left(R - 4(\nabla\Phi)^2 + (\nabla T)^2 + V(T) - 4 \right) \tag{25}$$

In the above equations we have set $\alpha' = 2$.

Classical Solutions:

In the absence of the tachyon field ($T = 0$) one can exactly solve (24).

The 1-parameter solution is given by

$$\begin{aligned} ds^2 &= G_{\mu\nu} dx^\mu dx^\nu = \frac{dudv}{uv + a} \\ e^{-2\Phi} &= uv + a \\ T &= 0 \end{aligned} \tag{26}$$

where $u = t + x, v = t - x$.

The above solution represents a 2-dim. black-hole,[24, 25] if $a > 0$ and then ‘a’ can be identified with the mass of the black-hole. One can compute the scalar curvature corresponding to the metric in (26),

$$R = \frac{4a}{uv + a} \tag{27}$$

The horizon is at $uv = 0$ and the curvature singularity is on the hyperbola $uv + a = 0$. When $a = 0$, (26) corresponds to (after a change of co-ordinates) to a flat space-time with a linear dilaton backgrounds,

$$\begin{aligned} G_{\mu\nu} &= \eta_{\mu\nu}, \quad \Phi = \frac{Q\eta}{2} \\ Q &= \sqrt{8} \end{aligned} \tag{28}$$

A few comments are in order:

- (i) The black hole solution (26) is remarkably similar to the Schwarzschild solution of 4-dim. general relativity. The hope is that its occurrence, in a ‘soluble model’ of string theory, may eventually shed some light on some of the important unsolved problems involving 4-dim. black-holes.

- (ii) The equation for tachyon propagation in the black hole is easily derived from (24) (neglecting non-linearities in the tachyon potential):

$$[4(uv + a) \partial_u \partial_v + 2(u \partial_u + v \partial_v) + 1] T(u, v) = 0. \quad (29)$$

This equation can be exactly solved but for our present purposes it is only necessary to note that the solution has a logarithmic singularity at the black-hole singularity:

$$T(u, v) \sim \ln(uv + a). \quad (30)$$

- (iii) Inclusion of higher order corrections in α' to the β -function does not change the essential nature of the black-hole solution.[26]
- (iv) The solution (28) is a starting point for a recursive solution of the full set (24). In fact (28) produces a tachyon background (neglecting non-linearities) $T_0(x) = (a + bx) e^{\frac{Q}{2}x}$. This in turn leads to a modification of the background (28) by terms of $o(e^{Qx})$ and so on. It turns out that tachyon scattering in the backgrounds $T_0(x) \sim o(e^{\frac{Qx}{2}})$ and $G_{0\mu\nu} - \eta_{\mu\nu} \sim o(e^{Qx})$, $\phi_0 - \frac{Q}{2}x \sim o(e^{Qx})$ reproduces the correct 3-point amplitude, say $1+2 \rightarrow 3$. [18] The physical picture is of wall scattering, the wall being provided by the above mentioned backgrounds. To compute a process like $1+2+3 \rightarrow 4$ (see eqn. (21)) in this method one would presumably not only have to solve for the backgrounds to higher orders in e^{Qx} but may also include the higher tensor fields of the string field theory. This technology is beyond present capabilities and hence even though the equations of motion provide the physical space-time picture, the matrix model is the only computational tool we have.
- (v) The β -function equations provide a natural framework to study black-hole formation and evaporation. However this has not been possible because of the extreme difficulty of solving these equations. CGHS[27] invented a simpler field theory model by simply replacing T in (24) by a conformal scalar field. In fact they introduced N such scalar fields to develop a semi-classical $\frac{1}{N}$ expansion for black-hole dynamics. Though we have learnt much along these lines it is fair to say that the basic issues involving black-hole dynamics remain unresolved in that model.

5. 2-dim. Black-Holes and the Matrix Model:

In the remaining part of this note we will focus on the possibility of discussing properties of black-holes in the 2-dim. target space using the matrix model.[28] For other approaches we refer to refs. (29) and (30).

Any attempt to understand the emergence of a non-trivial space-time in the matrix model has to contend with the fact that non-relativistic fermions are formulated in a flat space-time. However as we have seen the “space-time” in which the small perturbations on the fermi surface propagate is the half-plane with perfectly reflecting boundary conditions and local interactions. On the other hand since the target space theory has a metric $G_{\mu\nu}$ and a dilaton Φ , one can imagine an equivalent description of the system in which there is a field redefinition of the metric: $\tilde{G}_{\mu\nu} = G_{\mu\nu} e^{-2\Phi}$, which

corresponds to a space time which is flat, at least for the solution (27). Of course the field redefinition of the metric will imply in general a non-local and non-linear redefinition of the tachyon field.

In the following we present a transformation of the quantum phase space distribution of fermions, $\hat{u}(p, q, t)$ which was defined in eq. (7).

$$\hat{\phi}(p, q, t) = \int dp' dq' K(p, q|p', q') \hat{u}(p', q', t) \quad (31)$$

where

$$K(p, q|p', q') = |(p - p')^2 - (q - q')^2|^{-1/2} \quad (32)$$

The equation of motion for \hat{u} , that follows from the fermion equation of motion, is $\partial_t \hat{u} + p \partial_q \hat{u} + q \partial_p \hat{u} = 0$. One can verify that $\hat{\phi}$ satisfies the same equation and hence if we introduce the variables $u = \frac{1}{2} e^{-t}(p + q)$, $v = \frac{1}{2} e^t(p - q)$, we see that the equation of motion imply $\partial_t \hat{u}(ue^t + ve^{-t}, ue^t - ve^{-t}, t) = 0$ and $\partial_t \hat{\phi}(ue^t + ve^{-t}, ue^t - ve^{-t}, t) = 0$. Hence (31) effectively becomes a 2-dim. relation. Defining $\hat{T}(u, v) = \hat{\phi}(u + v, u - v, 0)$ we have

$$\begin{aligned} \hat{T}(u, v) &= \int du' dv' \tilde{K}(u, v|u', v') \hat{u}(u' + v', u' - v', 0) \\ \tilde{K} &= |(u - u')(v - v')|^{-1/2} \end{aligned} \quad (33)$$

Now if (33) has anything to do with black-holes it should have the property that for low energy scattering the background metric and dilaton perceived by the field $\hat{T}(u, v)$, correspond to the classical solution (27).

We demonstrate this in two steps: First, consider a state $|\psi\rangle$ in the fermion theory which differs from the classical ground state $|\psi_0\rangle$ so that $\delta u(p, q, t) = \langle \psi | \hat{u}(p, q, t) | \psi \rangle - \langle \psi_0 | \hat{u}(p, q, t) | \psi_0 \rangle$ has support, at most in a small neighbourhood of the fermi surface $p^2 - q^2 - 2\mu = 0$. Then $\delta T(u, v) = \langle \psi | \hat{T}(u, v) | \psi \rangle - \langle \psi_0 | \hat{T}(u, v) | \psi_0 \rangle$, is given by

$$\delta T(u, v) = \int du' dv' \hat{K}(u, v|u', v') \delta u(u', v') \quad (34)$$

The second step is that $\tilde{K}(u, v|u', v')$ has the following property:

$$\left[4 \left(uv - \frac{\mu}{2} \right) \partial_u \partial_v + 2(u \partial_u + v \partial_v) + 1 \right] \tilde{K}(u, v|u', v') = o \left(u'v' - \frac{\mu}{2} \right). \quad (35)$$

Now since $\delta u(u', v')$ has support in a small region around the fermi surface $u'v' = \frac{\mu}{2}$, (34) and (35) imply that

$$\left[4 \left(uv - \frac{\mu}{2} \right) \partial_u \partial_v + 2(u \partial_u + v \partial_v) + 1 \right] \delta T(u, v) = o \left(\frac{\delta E}{\mu} \right) \approx o \quad (36)$$

The differential operator on the l.h.s. in (36) is precisely the one that occurs in (29) with the identification $a = -\frac{\mu}{2}$. Hence the fermi level, the only dimensional parameter that specifies the ground state, is identified with the black-hole mass. δE in (36) is the maximum energy of the fermi fluid in the region deformed from the filled fermi sea. $\delta E/\mu$ is the expansion parameter proportional to the string coupling.

It is important to emphasize that even though (34) is a non-local linear functional of $\delta u(u, v)$, it is a non-local and non-linear function of the collective fields $p_{\pm}(q, t)$ that were introduced in (13). In fact one can explicitly express it as a power series in the fluctuations $\eta_{\pm}(q, t) = p_{\pm}(q, t) - p_{\pm}^0(q)$, $p_{\pm}^0(q) = \pm\sqrt{q^2 + 2\mu}$. This fact makes it a difficult enterprise to derive a closed equation for $\delta T(u, v)$.

6. The Question of Singularities in the Black-Hole Background

In the previous section we had stated that the tachyon field propagating in the black-hole background develops a singularity (eqns. (29),(30)), where the space-time curvature is singular. This would be one way of perceiving the black-hole singularity in an effective theory of tachyons. We now demonstrate that $\delta T(u, v)$ as defined in (34) using the fermi fluid theory has no singularity at $uv - \frac{\mu}{2} = 0$.

The issue is that of a 2-dim. integral of the form $\int dx dy f(x, y) |x^2 - y^2|^{-\frac{1}{2}}$. It is clear that such an expression is singular only if the function $f(x, y)$ is singular, because $|x^2 - y^2|^{-\frac{1}{2}} dx dy$ is regular at $x = y = 0$. Now in our theory $\delta u(u', v')$ is non-singular simply because it is a difference of 2 characteristic functions: $\delta u = u - u_0$. More generally, in the full quantum theory, a general state of the fermion theory $|\psi\rangle$ is obtained as a W_{∞} rotation of the fermion ground state $|\psi_0\rangle$. This implies that $\langle\psi|\hat{u}(p, q, t)|\psi\rangle$ is obtained as a W_{∞} rotation of the regular function $\langle\psi_0|\hat{u}(p, q, t)|\psi_0\rangle$, and hence $\langle\psi|\hat{u}(p, q, t)|\psi\rangle$ is a regular function on phase space.

The above general agreement can be supplemented by an explicit calculation in the case when $\delta u(u, v)$ is a simple local deformation of the fermi-surface. We simply quote the formulae from the literature.[28] Consider $\delta u(u, v)$ so that it is non-zero in a ‘small’ region around the fermi surface $uv = \frac{\mu}{2}$. Here by small we mean that the maximum energy of fermions in that region is given by $E = \mu + \Delta$, and $\Delta \ll |\mu|$. $\frac{\Delta}{\mu}$ is proportional to the string coupling. Then a calculation gives

$$\begin{aligned} \delta T(u, v) \simeq & \left(-\frac{\mu}{2}\right)^{-\frac{1}{2}} \left[\left(uv - \frac{\mu}{2}\right) \ln \left|uv - \frac{\mu}{2}\right| - \left(uv - \frac{\mu + \Delta}{2}\right) \ln \left|uv - \frac{\mu + \Delta}{2}\right| \right] \\ & + \text{(regular terms)} \end{aligned} \quad (37)$$

It is clear that $\delta T(u, v)$ has no singularities. It is regular at $uv = \frac{\mu}{2}$ and $uv = \frac{\mu + \Delta}{2}$. However an expansion of (37) in powers of $\frac{\Delta}{|\mu|}$ is divergent at $uv = \frac{\mu}{2}$ in every order of perturbation theory.

$$\Delta T(u, v) \simeq \left|\frac{\Delta}{\mu}\right| \ln \left|uv - \frac{\mu}{2}\right| + \left|\frac{\Delta}{\mu}\right|^2 \left(uv - \frac{\mu}{2}\right)^{-1} + o\left(\left|\frac{\Delta}{\mu}\right|^3\right) \quad (38)$$

The first term is the leading logarithmic divergence which we have already encountered in (30). It is clear that the non-linear completion of eqn. (29), that arises from string theory, cures the singularity if one sums (38) to all orders in perturbation theory. In that sense (37) is actually a non-perturbative result.

7.² Boundary condition and Tachyon Propagation in black-hole background[31]

The matrix model is exactly soluble; however the main difficulty in the subject is the space-time interpretation of the answers that emerge from the matrix model. Presently the only known way is to invent relevant transformations between quantities in the matrix model and 2-dim. string theory. Since the latter is mainly formulated in perturbation theory, we can compare the two theories at the classical level. This was the spirit behind the definitions (21) and (34).

The results that we presented about tachyon propagation, in a black-hole background in 2-dim. string theory via the matrix model made very minimal assumptions on the function $\delta u(u, v)$. We now discuss the boundary conditions on $\delta T(u, v)$, defined by (34), that correspond to various processes involving the scattering of tachyons by a black-hole. Firstly we note that $\delta T(u, v)$ defined by (34) is non-zero in the entire Kruskal plane $P = \{(u, v) | -\infty < (u, v) < +\infty\}$. In particular a generic $\delta u(u, v)$ gives, besides incoming flux from \mathcal{I}^- , a flux emerging from the white hole (see Fig.1). The demonstration of non-singular propagation would be more relevant for a more realistic collapse scenario if we could by some means avoid a flux of particles emanating from the line $v = 0$ in the Kruskal plane, and consider the half plane $P_+ = \{(u, v) | v \geq 0\}$ as the physical space-time along with the boundary condition $\delta T(u, v = 0) = 0$.

A sufficient condition that achieves this boundary condition is

$$\int_{-\infty}^{\infty} dv' |v'|^{-\frac{1}{2}} \delta u(u', v') = 0 \quad (39)$$

We will show that for a very large class of fluctuations $\delta u_1(u, v)$ it is possible to construct a modified fluctuation $\delta u(u, v) = \delta u_1(u, v) - \delta u_2(u, v)$ (where $\delta u_2(u, v)$ depends on $\delta u_1(u, v)$) which satisfies (39). We can use this result to generate infinitely many solutions of (39).

We will show the result (easily generalizable to other cases) for $\delta u_1(u, v)$ which consists of one “blip” bulge in the fermi surface, and one “antiblip” (dip in the fermi surface). Further we will assume that the modified fermi surface has a “quadratic profile”. This means, e.g., that the blip is described by the formula

$$\delta u_1(u, v) = \theta [(v_+(u) - u) (u - v_-(u))] . \quad (40)$$

between some u_1 and u_{\max} (see Fig. 2). In eq. (40) and Fig. 2 we have ignored the antiblip which can be discussed similarly. The figure shows that $v_+(u) = v_+^{(0)}(u) = \frac{\mu}{2u}$ for $u \in [u_1, u_2]$. Let us consider a fluctuation

$$\delta u_2(u, v) = \theta [(\tilde{v}_+(u) - v) (v - \tilde{v}_-(u))]$$

for $u \in [u, u_{\max}]$ and zero elsewhere. Clearly (39) can be satisfied by $\delta u(u, v)$ if

$$\begin{aligned} \sqrt{\tilde{v}_+(u)} &= \sqrt{v_+(u)} - f(u) \\ \sqrt{\tilde{v}_-(u)} &= \sqrt{v_-(u)} + f(u) \end{aligned}$$

²This section has been added for completeness in the proceedings and was not part of the talk.

where $f(u)$ is arbitrary and positive in the region $[u_2, u_{\max}]$ and vanishes elsewhere.

Note that $\delta u_1(u, v)$ and $\delta u_2(u, v)$ can, like before, be chosen close enough to the fermi surface so that eq. (36) is again valid. Note that $\delta u(u, v)$ that we have constructed is not a “quadratic profile” and in the range (u_2, u_{\max}) , a $u = \text{constant}$ line intersects it in 4-points.

The point of the above demonstration is that if we use fermi fluid distributions that eliminate the particle flux from the white hole, then our previous demonstration of the absence of singularity in $\delta T(u, v)$ is more relevant to the more physical collapse scenarios of black-hole dynamics. Solutions like the one described above correspond to nonnormalizable wave packets at \mathcal{I}^- and \mathcal{I}^+ .

We also want to state that collapse scenarios for black holes are bound to be different in case the underlying theory (like in the case of 2-dim. string theory) has an infinite number of conserved charges.

Concluding Remarks:

We have presented a certain view of 2-dim. string theory. If one takes stock of the achievements it is fair to say that the matrix model and the leg pole prescription enables in principle a calculation of the S -matrix for wall scattering to all orders in perturbation theory. An explicit demonstration of higher order corrections to the S -matrix using the quantum (conserved) W_∞ charges would be desirable.

Regarding non-perturbative effects, we could demonstrate that the absence of a singularity in the tachyon wave at the black hole singularity was indeed a non-perturbative effect. This is because one needs to sum a series each term of which is divergent at every order of the semi-classical expansion, at the black hole singularity, but the full sum is finite and singularity free.

Regarding open questions we would list the following:

- (i) We do not yet know what the stringy non-perturbative effects[32] $\left(\sim e^{-\frac{1}{g_{str}}} \right)$ means in the target space of 2-dim. string theory. In a manner of speaking we know the answer but not the question.
- (ii) The same comment also applies to the strongly coupled 2-dim. stringy theory. In the matrix model this means $\mu = \frac{1}{g_{str}} \rightarrow 0$, and the fermi level is very near the tip of the inverted harmonic oscillator potential. Clearly the picture of “wall scattering” is not applicable anymore as the fermi fluid can easily trickle to the ‘other side’ of the potential. We do not know the ‘strong coupling’ question in 2-dim. string theory. The situation here is more difficult than it was in gauge theories in the 1970s, because at that time lattice gauge theorists had a phenomenological picture of quark confinement that they wanted to explain: The squeezing of chromo-electric flux between a quark and anti-quark. It would be interesting to know the corresponding questions in string theory in two or for that matter even in four dimensions.
- (iii) It is conceivable that the β -function equations (24) and their quantization can describe the formation and evaporation of black holes in 2-dim. string theory. It

would be of great interest to know how these processes can be described in the matrix model.

Acknowledgement:

The ‘view’ of 2-dim. string theory and black hole physics presented here was mainly developed with Avinash Dhar and Gautam Mandal and also in the early stages with Sumit Das and Anirvan Sengupta. I have greatly enjoyed the experience and I thank them for it. I would like to thank Gautam Mandal for a careful reading of the manuscript.

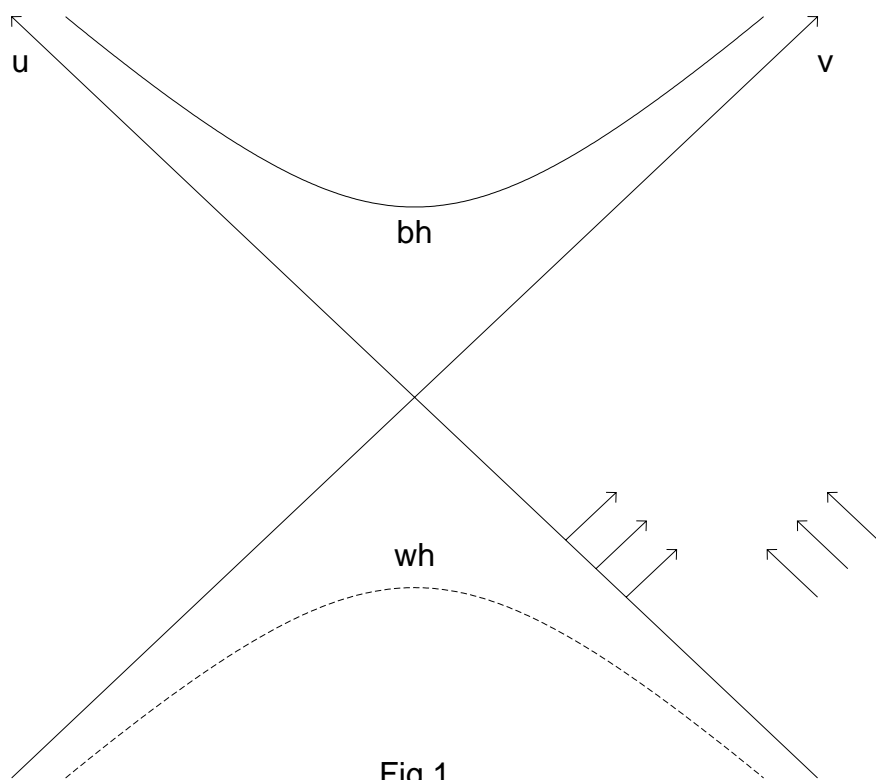
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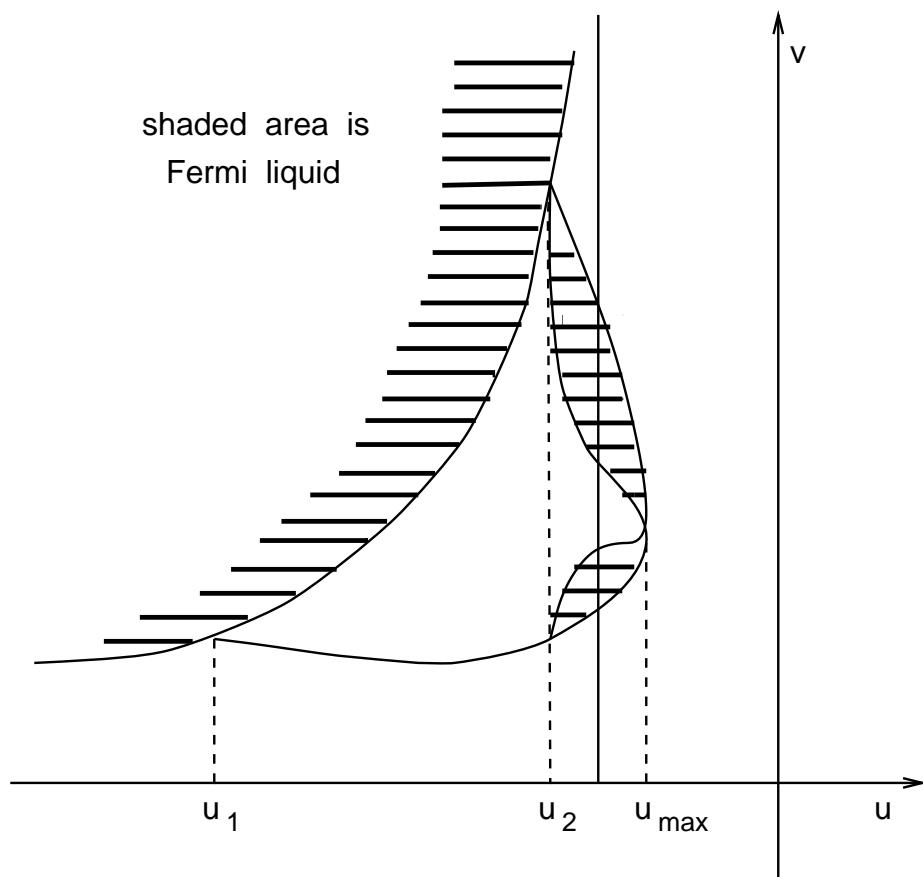


Fig 2